UNDER THE HOOD OF A CARBON FOOTPRINT CALCULATOR

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Abstract. We explain the mathematical theory of the Input-Output method for carbon footprints computations.

Introduction

Two main methods for calculating carbon footprints are the *Input-Output* (IO) method and the *Lifecycle Analysis* (LCA) method, also known as *Craddle-to-Grave*. Both aim to quantify the greenhouse gas (GHG) emissions of a product or service, but they differ in scope, data needs, and level of detail.

The Input-Output method is an economic approach developed in [?], that uses national or regional data on industries and their interdependencies. It calculates the carbon footprint by tracing the flow of goods and services across different sectors, estimating emissions based on economic input-output tables. It relies on aggregated data about economic sectors (e.g., agriculture, manufacturing) and their emissions, and estimates emissions associated with production, supply chains, and services by analyzing the economic connections between industries. This is the commonly accepted method for large-scale, economy-wide assessments is not data-intensive, relies on publicly available economic data. It will lack detail on specific products or processes and miss the emissions from smaller, niche activities.

The Lifecycle Analysis evaluates a product's environmental impact over its entire lifecycle — from raw material extraction to disposal. It calculates emissions at each stage, considering both direct and indirect impacts. This provides a detailed, product-specific carbon footprint, identifies emissions hotspots and opportunities for reduction. It is however data-intensive and time-consuming, and the total complexity can lead to global inconsistencies, as the under or over counting of secondary emissions are only estimated.

The main difficulty in the concrete application of the Input-Output method is access to the relevant data: a recent Input-Output table (or

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transaction matrix) and a measure of environmental impact for each sector appearing in the table. The tables are quite available, see for Switzerland [**?**]. The latter is harder to obtain, especially as it needs to be computed/collected in an homogeneous manner over the different sectors in order to make sense.

This note aims to explain the mathematical theory of the Input-Output method, and is based on the article by J. Kitzes [**?**], the talk by J. Steinberger at the Climathics 2022 conference [**?**], and the third author's talk for the *Comission Romande de mathématiques* in 2023.

1. A concrete question

To estimate the carbon footprint of a T-shirt for instance, one needs the footprint of the cotton all the way from production of the raw material up to the stitching of the T-shirt, all the washing during the lifespan of the T-shirt, and the disposal of the unusable T-shirt. But to estimate the cotton footprint only, one needs to estimate the transport of the water, the one of the pesticides used to protect the crop, the fertilizing of the field, all those requiring tractors and fuel, the fuel itself requiring transport and fuel, thus creating loops that are difficult to extricate. Moreover, once one estimates all the emissions, there's no guarantee that all the emissions of all the objects that we so compute, will actually sum up to all the emissions that we actually produce. The Input-Output method takes the problem from the other end: it takes *all* the measured emissions, and then attributes it to different sectors, and then computes the carbon footprint of a goods from the textile industry according to its direct emissions, combined with all the indirect emissions as we shall now explain.

2. Basic Input-Output

This method models a closed economy with n *economical sectors*, $S = (S_1, \dots, S_n)$, and comes in the following form:

	S_1	S_2	$\dots$	S_n	D	T
S_1	c_{11}	c_{12}	$\dots$	c_{1n}	d_1	t_1
S_2	c_{21}	c_{22}	$\dots$	c_{2n}	d_2	t_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
S_n	c_{n1}	c_{n2}	$\dots$	c_{nn}	d_n	t_n
V	v_1	v_2	$\dots$	v_n		
T	t_1	t_2	$\dots$	t_n		

The $n \times n$ *transaction matrix* is $C = (c_{ij}) \in M_n(\mathbf{R}_+)$ (positive coefficients only) has entry c_{ij} the number representing the economical transactions from sector S_i to S_j. The diagonal entries c_{ii} then represent the internal gross revenue of the sector. The *demand vector* $D = (d_1, \dots, d_n)$ expresses the money spent by the population on each sector: d_i is the amount bought from sector S_i, for $i = 1, \dots, n$. The *added value* vector is given by $V = (v_1, \dots, v_n)$ and the *total* vector $T = (t_1, \dots, t_n)$ satisfy the following relations:

$$t_i = \sum_{j=1}^{n} c_{ij} + d_i = v_i + \sum_{j=1}^{n} c_{ji}$$

for $i = 1, \dots, n$. The value t_i represents the *total output* of the sector S_i, which is also the *total input* because the system is a closed circuit. Out of these vectors we define the *technical coefficient matrix* $A = (a_{ij})$ where

$$a_{ij} = \frac{c_{ij}}{t_j}.$$

For a vector $W = (w_1, \dots, w_n)$ we denote by D_W the diagonal matrix with entries $w_1, \dots, w_n$ in the diagonal, so that $A = CD_T^{-1}$. The scalar product of two vectors $X = (xi, \dots, x_n)$ and $Y^t = (y_1, \dots, y_n)$ is given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = XY$$

where X is a *line vectors*, namely $1 \times n$ matrix, and Y is a *column vector*, namely an $n \times 1$ matrix and XY is the usual matrix multiplication so a real number.

3. Footprints in the Input-Output model

Economical activities have all sorts of footprints that are correlated to the income and the cost of things. To compute the footprint we need to have a *emission* vector $E = (e_1, \dots, e_n)$ where e_i is the direct emission of sector S_i. It could be millions of tons of CO_2, measured or estimated directly above the industries of the sector, or square feet measured by the occupancy of the industries, or tons of plastic trash estimated by the collectivities, etc. Then $|E| = e_1 + \cdots + e_n$ is the total footprint of the economical system. We compute the *direct intensity* of the print by $F = (f_1, \dots, f_n)$ with

$$f_i = \frac{e_i}{t_i}$$

each entry is the print per unit of money. Now, if we compute the inner product

$$\langle D, F \rangle = \sum_{i=1}^{n} d_i f_i$$

it is the footprint directly attributed to the consumers, but it doesn't take into account all the intermediate exchanges that the sectors had to create the product. If you buy goods for k dollars to industry S_i, then kf_i indicates the direct footprint of the purchase, but not the total footprint of the goods. That one has to take into account the exchanges the sector S_i had from the other ones. To do this, we need to compute the *total intensity vector* $X = (x_1, \cdots, x_n)$ that gives the total footprint of the sector S_i by money amount of final consumer demand, given by

$$X = F + FA + FA^2 + \cdots = F(I - A)^{-1}$$

where I denotes the $n \times n$ identity matrix (most matrices are invertible so let's assume that one is).

Proposition 1. *The footprint is completely attributed to the consumers. Namely, with the above notations, $\langle X, D \rangle = |E|$.*

Proof. First notice that the equations $t_i = \sum_{j=1}^{n} c_{ij} + d_i$ for $i = 1, \ldots, n$ rephrase as $D = T - C\mathbf{1} = T - AD_T\mathbf{1} = T - AT = (I - A)T$, where $\mathbf{1} = (1, \ldots, 1)$ and T is the total vector and D the demand vector, both viewed as column vectors. We compute:

$$\langle X, D \rangle = \left\langle F(I - A)^{-1}, (I - A)T \right\rangle = FT = \sum_{i=1}^{n} \frac{e_i}{t_i} t_i = |E|$$

□

Remark 1. Similarly one can define $Y = F(I - A^t)^{-1}$ such that $\langle Y, V \rangle = |E|$ and that vector attributes the print to the added value of the different sector.

4. Limitations

Waste. In a closed economy, in order to take into account waste disposal, one needs a sector S_i that runs the waste management. That sector has a large footprint since incinerators produce large quantities of CO_2 (of the order of 1kg CO_2 per kg of waste incinerated [**?**]), and is eventually attributed to the consumers through waste management taxes (in France the weight of waste is estimated at half a ton per person and per year). The model also only doesn't discuss demand and waste, assuming that everything is bought and nothing gets wasted.

Added value. The computation doesn't take into account the added value V, which is what the companies make, and the model attributes the footprint completely to the customers, through the vector D. Attributing the footprint to the added value created by the sectors would seem more natural to understand their responsibility, see Remark **??** above.

Inverses. Mathematically speaking, the main problem is that if B is a matrix ϵ-close to A (say in terms of the coefficients), then the distance between $(I-A)^{-1}$ and $(I-B)^{-1}$ could be a priori be as big as we want, for instance if the matrices both have an eigenvalue close enough to 1. Since all the values are estimated, the model is probably not very stable in an optimization exercise.

Life Cycle Analysis (LCA). To get a comprehensive and detailed view of the total carbon footprint of a given goods, one can examine each stage, including production, transportation, usage, and end-of-life processes. This method can estimate the footprint of goods with a detail that Input-Output can't, but uses data-intensive estimates that need to be computed, with not total consistency guarantee whatsoever, and no way to estimate the inconsistency.

References

[Clim] https://www.la-matrice.ch/climathics-workshop

[JK] Justin Kitzes *An Introduction to Environmentally-Extended Input-Output Analysis*. Resources 2013, 2, 489-503; doi:10.3390/resources2040489

[L] Leontief W., *Input-Output Economics*, Oxford University Press, Oxford, 1966

[IOCH] https://www.bfs.admin.ch/bfs/en/home/statistics/national-economy/input-output.html

[Waste] Municipal waste statistics, available from https://ec.europa.eu/eurostat/statistics-explained/index.php